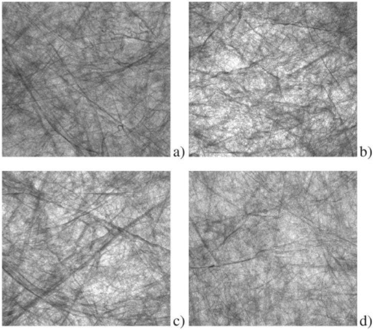
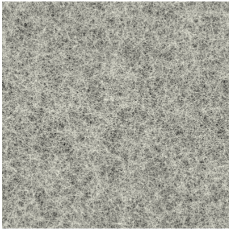
Contents

(VER training session)

Training session

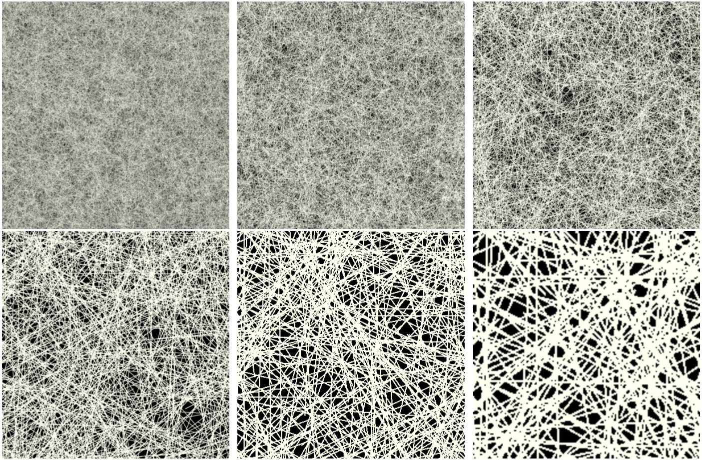
Microscopy images from Godehardt et al (2022) + a realization of a random set model





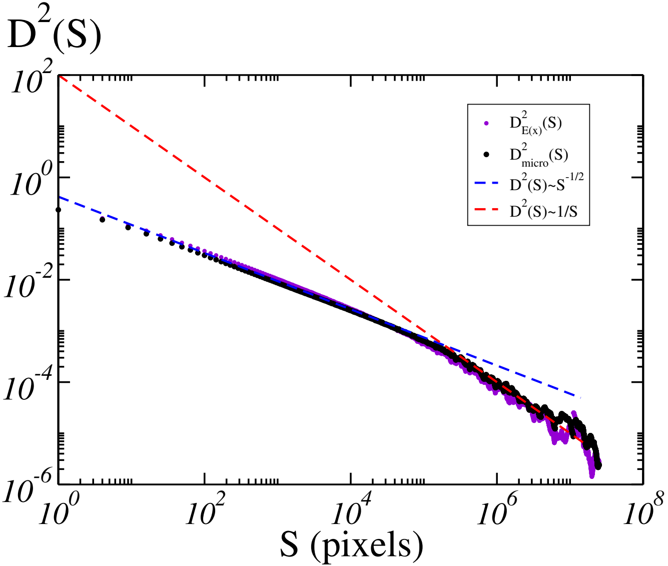
Training session

Random set at different length scales



Training session

Variances scaling law. Boolean model of fibres in 2D.



Training session

A general result gives a complete characterization of the Choquet capacity for Boolean models :

$$T(K) = P\{X \cap K \neq \varnothing\} = 1 - \mathrm{e}^{-t|R_0 \oplus \check{K}|}$$

where $\check{K} = \{-\boldsymbol{x} | \boldsymbol{x} \in K\}$, and $\oplus$ is the Minkowski addition :

$$R_0 \oplus \check{K} = \{\boldsymbol{x} | K_{\boldsymbol{x}} \cap R_0 \neq \varnothing\} = \bigcup_{x \in \check{K}} R_{0x}$$

with $K_x = \{\boldsymbol{x} + \boldsymbol{y} | \boldsymbol{y} \in K\}$.
Example : take $K = \{x\}$, then :

$$p = T(K) = 1 - \mathrm{e}^{-t|R_0 \oplus \check{K}|} = 1 - \mathrm{e}^{-tab}.$$

The value p represents the density of X.

$$K \subset X^c \Leftrightarrow x \in (X^c \ominus \check{K}) \Leftrightarrow x \in (X \oplus \check{K})^c = \bigcup_{x \sim \mathcal{P}(t), \theta\ \mathcal{U}(0,2\pi)} \{R_x(\theta) \oplus \check{K}\}^c$$

Training session

Boolean model of fibres in 2D. Covariance ($b \gg h \gg a$) :

$$C(h) = 2p - 1 + (1-p)^{2-k(h)}$$

where :

$$k(h) = \langle K(h,\theta) \rangle_{0 \leqslant \theta \leqslant \pi} = \frac{2}{\pi} \int_0^{\min\{\pi/2, \sin^{-1}(a/h)\}} \mathrm{d}\theta \, K(h,\theta)$$

is the mean covariogram and :

$$K(h,\theta) = \max(0, b - h\cos\theta) \max(0, a - h\sin\theta)$$

is the variogram of a rectangle of length a, b.

Mean covariogram ($b > a$) :

$$k(h) = \frac{2}{\pi} 1_{h \leqslant b} \int_0^{\min\{\pi/2, \sin^{-1}(a/h)\}} \mathrm{d}\theta \, (b - h\cos\theta)(a - h\sin\theta)$$

$$= \frac{2}{\pi} 1_{h \leqslant b} \times \begin{cases} \int_0^{\pi/2} \mathrm{d}\theta \, (b - h\cos\theta)(a - h\sin\theta) & \text{if } h \leqslant a \\ \int_0^{\sin^{-1}(a/h)} \mathrm{d}\theta \, (b - h\cos\theta)(a - h\sin\theta) & \text{if } h > a \end{cases}$$

If $b \gg h \gg a$:

$$k(h) \approx \frac{a^2 b}{\pi h} 1_{h \leqslant b}$$

Training session

Boolean model of fibres in 2D.

Covariance ($b \gg h \gg a$) :

$$C(h) = 2p - 1 + (1-p)^{2-k(h)} = p^2 + (1-p)^2 \left[(1-p)^{-k(h)/k(0)} - 1 \right]$$

$$\frac{C(h) - p^2}{p(1-p)} \approx \left(\frac{1}{p} - 1 \right) \left[(1-p)^{-a/(h\pi)} - 1 \right]$$
$$= -\log(1-p) \left(\frac{1}{p} - 1 \right) \frac{a}{h\pi}$$

Integral range : $A_2 = +\infty$

Training session

Asymptotic expansion of the integral range. Integral computed on a disc of radius $L \to \infty$:

$$\int_{0 \leqslant \ell \leqslant L, 0 \leqslant \theta \leqslant 2\pi} \mathrm{d}\ell \, \ell \mathrm{d}\theta \frac{C(\ell) - p^2}{p(1-p)}$$

$$\approx -2\pi \log(1-p) \left(\frac{1}{p} - 1 \right) \int_{0 \leqslant \ell \leqslant L} \mathrm{d}\ell \frac{a}{\pi}$$

$$= -2a \log(1-p) L \left(\frac{1}{p} - 1 \right).$$

Variance on a ball of radius L with $a \ll L \ll b$:

$$D^2(S) = -\frac{2a}{\pi L}(1-p)^2 \log(1-p) + o(1/L^2) \sim \frac{-2a(1-p)^2 \log(1-p)}{\sqrt{\pi S}}$$

Training session

Boolean model of cylinders in 3D (Gille, 1989)

$$A_3(\ell) = \int_{t\leq\ell} (4\pi t^2)\mathrm{d}t\,\left[C(t) - p^2\right] \sim \begin{cases} -\frac{2\pi(1-p)}{p} r^2 \log(1-p)\ell, & \text{if } h \to \infty, \\ -\frac{\pi(1-p)}{p} h \log(1-p)\ell^2, & \text{if } r \to \infty. \end{cases}$$

$$D^2_{\mathbf{B}}(V) \sim \frac{-9(1-p)^2 r^2 \log(1-p)}{8\ell^2} + O(1/\ell)^3, \qquad (h = +\infty),$$

$$D^2_{\mathbf{B}}(V) \sim \frac{-3h(1-p)^2 \log(1-p)}{5\ell} + \frac{9h^2(1-p)^2\left[\log(1-p)\right]^2}{32\ell^2} + O(1/\ell)^3, \qquad (r = +\infty).$$

$$D^2_{\mathbf{B}}(V) \sim \frac{-27(1-p)^2 \log(1-p) h r^2}{8\ell^3}, \qquad (\ell \gg h \gg r),$$

$$D^2_{\mathbf{B}}(V) \sim \frac{-3(1-p)^2 \log(1-p) h r^2}{5\ell^3}, \qquad (\ell \gg r \gg h).$$

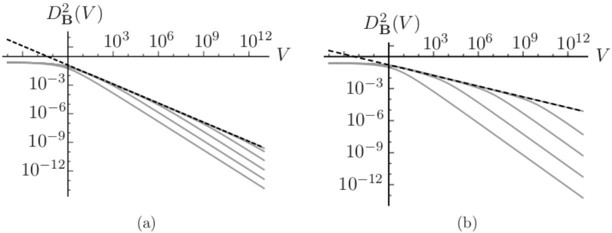


Figure 6: Variance $D^2_{\mathbf{B}}(V)$ of the estimate of the volume fraction measured in a spherical domain W vs. volume size V of the domain, in log-log plot. The cylinders volume fration is fixed to $p = 1/2$. Grey solid lines, bottom to top : **(a)** cylinders height $h = 1,\ 10,\ 100,\ 10^3,\ 10^4,\ \infty$ with radius fixed to $r = 1/2$; **(b)** cylinders radius $r = 1,\ 10,\ 100,\ 10^3,\ \infty$ with height fixed to $h = 1$. Dotted lines in black: expansions (45).