The RVE method
for random sets and homogenization problems
Training Session

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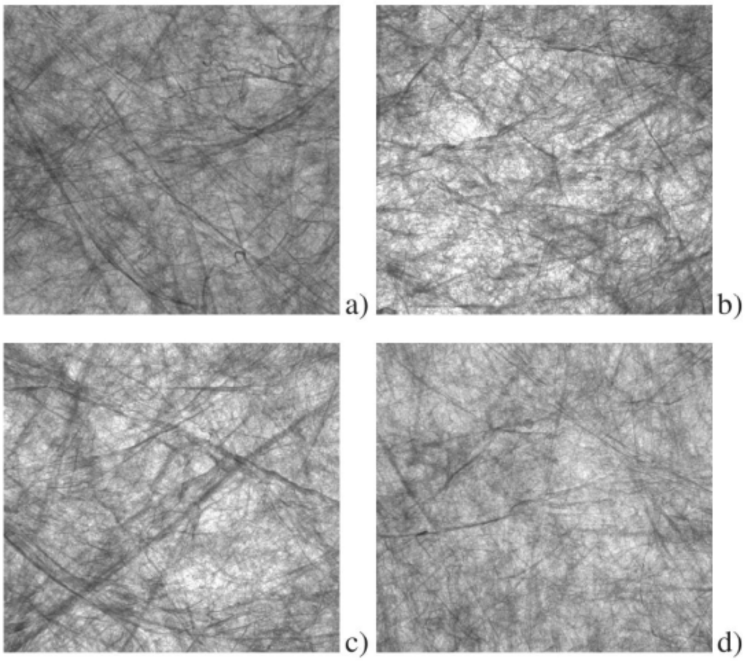
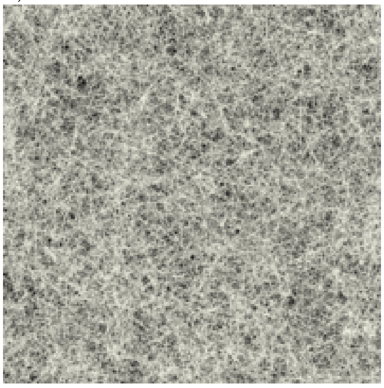
1 Random function

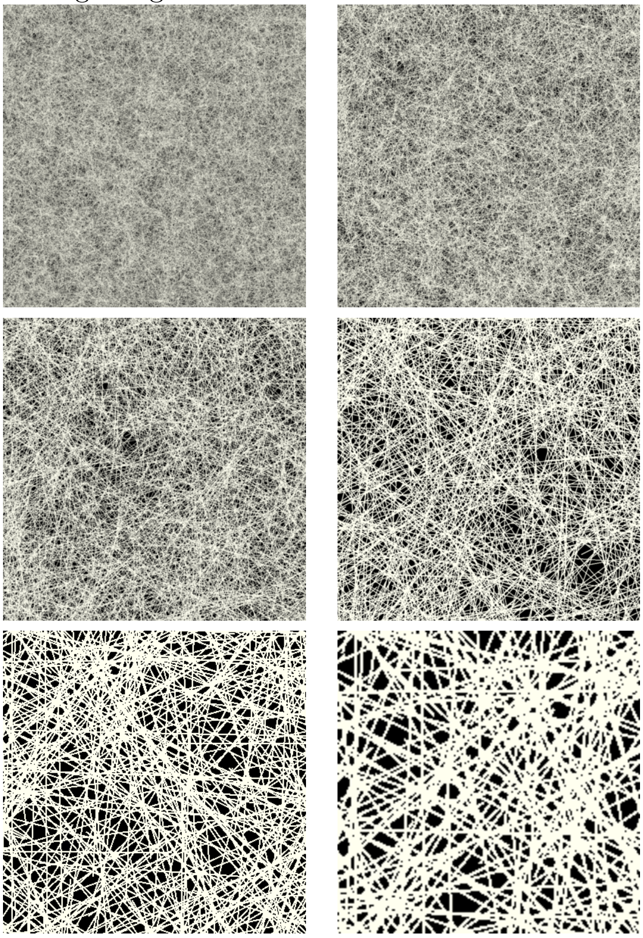
Observe the following images or a real material, taken from a recently-published study by Fraunhofer ITWM. Try to guess which material this might be.



Observe the following image, produced numerically using a probabilistic model. Try to guess how this random set is defined (answer on the next page).



The image in the previous page is a 2D Boolean model of rectangles, seen at large length-scale. Here how it looks when we zoom in.



2 Boolean model

More precisely, the above random set is defined as the union of uniformly-oriented rectangles:

$$X = \bigcup_{x\sim\mathcal{P}(t),\theta\ \mathcal{U}(0,2\pi)} R_x(\theta)$$

where the $R_x(\theta) = x + y|y \in R_0(\theta)$ are translations by x of the rectangle $R_0(\theta)$ and $R_0(\theta)$ is the rotation by theta of a fixed rectangle R_0 with length $a < b$. Furthermore $\mathcal{U}(0, 2\pi)$ is the uniform distribution in $[0; 2\pi]$ and $\mathcal{P}(t)$ is a Poisson point process of intensity t.

A Poisson point process is a random process of points in space. By definition, it is a Poisson point process iff

- The number of points $N(\Omega)$ that fall in a domain Ω follows the Poisson law:

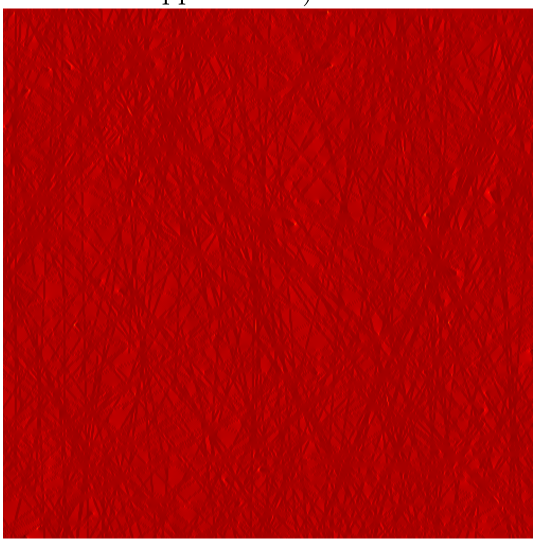
$$P\{N(\Omega) = n\} = \exp\left(-t|\Omega|\right) \frac{(t|\Omega|)^n}{n!}$$

- $N(\Omega)$ and $N(\Omega')$ are independant random variables if Ω and Ω' are disjoint.

The first condition follows from the second one if t is the mean number of points per unit of surface, and the Poisson process is homogeneous. So Boolean realizations are achieved by drawing a number $n \sim N(\Omega)$ according to the above Poisson law, and implanting n grains at random poisitions in Ω, with the same probability in each surface element δS.

3 Variance scaling laws

The files "varMicro.txt" and "varEx.txt" provides the variance of the mean of the characteristic function of X over windows of size V and that of the electric field in the heterogeneous medium X. The electric field has been computed by FFT assuming the fibers are highly-conducting. Below's an enlargement of a region for the electric field in the medium (component parallel to the applied field).



Plot in log-log plot the data present in "varMicro.txt" and "varEx.txt". The volume V is in column 1 (in pixels) and the field variance in column 2. Compare this data with theoretical predictions, such as the asymptotic expression provided by Matheron for stationary fields. Comment on the result. Can you abserve additional scaling laws other than that predicted by the theory? In which regime do they appear?

4 RVE for the Boolean model

A general result gives a complete characterization of the Choquet capacity for Boolean models:

$$T(K) = P\{X \cap K \neq \varnothing\} = 1 - \mathrm{e}^{-t|R_0 \oplus \check{K}|} \tag{1}$$

where $\check{K} = \{-\boldsymbol{x}|\boldsymbol{x} \in K\}$, and $\oplus$ is the Minkowski addition:

$$R_0 \oplus \check{K} = \{\boldsymbol{x}|K_{\boldsymbol{x}} \cap R_0 \neq \varnothing\} = \bigcup_{x \in \check{K}} R_{0x}$$

with $K_x = \{\boldsymbol{x} + \boldsymbol{y}|\boldsymbol{y} \in K\}$.

Example: take $K = \{x\}$, then:

$$p = T(K) = 1 - \mathrm{e}^{-t|R_0 \oplus \check{K}|} = 1 - \mathrm{e}^{-tab}.$$

The value p represents the density of X. Prove the above. Hint: consider the variation δp that result from adding some grains/rectangles).

(Subsidiary question) Prove (1) using the identity:

$$K \subset X^c \Leftrightarrow x \in (X^c \ominus \check{K}) \Leftrightarrow x \in (X \oplus \check{K})^c = \bigcup \{R_x(\theta) \oplus \check{K}, x \sim \mathcal{P}(t), \theta\, \mathcal{U}(0, 2\pi)\}^c$$

Let us now compute the Choquet capacity when $K = \{x, x+h\}$, equal to the covariance function $C(h)$. The resulting expression depends on the mean variogram:

$$k(h) = \langle |R_0 \cap R_{0\boldsymbol{h}}| \rangle_{|\boldsymbol{h}|=h}$$

given by the surface of the intersection of R_0 with the same rectangle translated by a vector of length h, averaged over all orientations. this provides us with:

$$C(h) = 2p - 1 + (1-p)^{2-k(h)/k(0)}$$

(Subsidiary question:) prove the above. Show also that $C(h) = P\{\boldsymbol{x} \in X, \boldsymbol{x} + \boldsymbol{h} \in X, |\boldsymbol{h}| = h\}$.

Compute now the limiting behavior of $k(h)$ as $h \to \infty$, but sill assuming $h \ll b$ (so that $b \to \infty$).

Same question for $C(h)$.

Compute the integral range A_2 using the expression for $C(h)$ in the range $a \ll h \ll b$. We recall that:

$$A_d = \int_{h \in \mathbb{R}^d} \mathrm{d}^d \boldsymbol{h} \frac{C(\boldsymbol{h}) - C(0)^2}{C(0)[1 - C(0)]}.$$

Compute the limiting behavior of the pseudo integral range (integrated over a disk of radius L):

$$A_2(L) = \int_{0 \leqslant \ell \leqslant L, 0 \leqslant \theta \leqslant 2\pi} \mathrm{d}\ell\, \ell \mathrm{d}\theta \frac{C(\ell) - p^2}{p(1-p)}$$

in the domain $a \ll h \ll b$

Compute the pseudo-variance:

$$D^2(S; L) = -\frac{\text{var}(X) A_2(L)}{S}$$

Show that the above gives the variance of the field in disks of radius L with $a \ll L \ll b$ as $b, L \to \infty$. Interpret the analytical and numerical results.